

Perturbation theorems for Hele-Shaw flows and their applications

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Abstract

In this work, we give a perturbation theorem for strong polynomial solutions to the zero surface tension Hele-Shaw equation driven by injection or suction, so called the Polubarinova-Galin equation. This theorem enables us to explore properties of solutions with initial functions close to but are not polynomial. Applications of this theorem are given in the suction or injection case. In the former case, we show that if the initial domain is close to a disk, most of fluid will be sucked before the strong solution blows up. In the later case, we obtain precise large-time rescaling behaviors for large data to Hele-Shaw flows in terms of invariant Richardson complex moments. This rescaling behavior result generalizes a recent result regarding large-time rescaling behavior for small data in terms of moments. As a byproduct of a theorem in this paper, a short proof of existence and uniqueness of strong solutions to the Polubarinova-Galin equation is given.

Keywords: Hele-Shaw flows, starlike function, rescaling behavior.

1 Introduction

This paper deals with classical zero surface tension (ZST) Hele-Shaw flows. The driving mechanism, injection or suction with a constant rate 2π or -2π at the origin, produces a family of domains $\{\Omega(t)\}_{t \geq 0}$. In two dimensions, Galin and Polubarinova-Kochina reformulated the planar model of Hele-Shaw flows by describing the domains $\{\Omega(t)\}$ by a family of conformal mappings $\{f(\xi, t)\}$ where $f(\xi, t) : D \rightarrow \Omega(t)$ and $f(0, t) = 0, f'(0, t) > 0$. Here we set

$$f_t(\xi, t) = \frac{\partial}{\partial t} f(\xi, t), \quad f'(\xi, t) = \frac{\partial}{\partial \xi} f(\xi, t), \quad D = D_1(0), \quad D_r = D_r(0)$$

where $D_r(z_0) = \{x \in \mathbb{R}^2 : |x - z_0| < r\}$. Equations for $f(\xi, t)$, so called the Polubarinova-Galin equations, are derived under this reformulation and they

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are expressed in the case of injection or suction respectively as follows:

$$\operatorname{Re} \left[f_t(\xi, t) \overline{f'(\xi, t) \xi} \right] = 1, \quad \xi \in \partial D \quad (1.1)$$

and

$$\operatorname{Re} \left[f_t(\xi, t) \overline{f'(\xi, t) \xi} \right] = -1, \quad \xi \in \partial D. \quad (1.2)$$

A solution to equation (1.1) or (1.2) is said to be a strong solution for $t \in [0, b]$ if $f(\xi, t)$ is univalent and analytic in a neighborhood of \overline{D} , $f(0, t) = 0$, $f'(0, t) > 0$ and $f(\xi, t)$ is continuously differentiable in $t \in [0, b]$.

Denote

$$\begin{aligned} H(E) &= \{f \mid f(\xi) \text{ is analytic in } E\}, \\ O(E) &= \left\{ f \in H(E) \mid f(\xi) \text{ is univalent, } f(0) = 0, f'(0) > 0 \right\}. \end{aligned}$$

The short-time well-posedness of (1.1) has been thoroughly explored. In Reissig and von Wolfersdorf [7], the authors prove the existence and uniqueness of a short-time strong solution in $O(\overline{D})$ if the initial function is in $O(\overline{D})$. In Gustafsson [1], the author proves that a strong solution to (1.1) is a family of polynomials of degree k_0 if its initial function in $O(\overline{D})$ is also a polynomial of degree k_0 . These results all can be applied to (1.2) as well even though the authors don't comment on that.

In this paper, we first prove a perturbation theorem for the strong polynomial solutions to the Polubarinova-Galin equation (1.1) or (1.2). Many properties for strong polynomial solutions are thoroughly known. This theorem enables us to explore the properties of evolution of perturbed polynomials which are nonpolynomial. We obtain two applications of this theorem in the suction and injection case.

We first state this perturbation theorem. We define the following norms to describe the evolution of solutions:

$$\left| \sum_{i=0}^{\infty} a_i \xi^i \right|_M = \sum_{i=0}^{\infty} |a_i|, \quad \left| \sum_{i=0}^{\infty} a_i \xi^i \right|_{M(r)} = \sum_{i=0}^{\infty} |a_i r^i|.$$

Also, we define the following norm to describe the small perturbation:

$$\|v\|_{\rho, n} = \sum_{j=1}^{\infty} |v_j| \rho^j j^{\frac{1}{2}+n}, \quad v = \sum_{j=1}^{\infty} v_j \xi^j.$$

The perturbation theorem, Theorem 1.1, describes the evolution of small perturbation of polynomials and is stated as follows:

Theorem 1.1. *Given a strong degree k_0 polynomial solution $f_{k_0}(\xi, t)$ to (1.1) (or (1.2)), and that $f_{k_0}(\xi, t) \in O(\overline{D}_r)$ at $t \in [0, T_0]$ for some $T_0 > 0$ and $r > 1$. Then for $\epsilon > 0, k \in \mathbb{N}$ and $1 < r' < r$, there exist $\delta(f_{k_0}, T_0, \epsilon, k, r') > 0$ and*

$\rho(f_{k_0}, T_0, \epsilon, k, r') > 1$ such that if $\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\rho, k} < \delta$ where $f(0, 0) = 0$ and $f'(0, 0) > 0$, then the strong solution to (1.1) (or (1.2)) $f(\xi, t)$ satisfies

$$f(\xi, t) \in O(\overline{D_{r'}}) \cap C^1([0, T_0], H(D_r)),$$

and for $0 \leq n \leq k$, $0 \leq t \leq T_0$,

$$\left| f_{k_0}^{(n)}(\cdot, t) - f^{(n)}(\cdot, t) \right|_{M(r)} < \epsilon.$$

The applications of this theorem and related past results are stated briefly in 1.1 and 1.2 as the following:

1.1 Here we assume the driving mechanism is suction. It has been known that strong solutions to (1.2) must blow up before the fluid is sucked out except for the degree 1 polynomial solutions. However, by taking $k_0 = 1$ in Theorem 1.1, we prove that if the initial domain is close to a disk, most of fluid is sucked before the strong solution to (1.2) blows up.

1.2 Now we assume the driving mechanism is injection. In Sakai [10] and Gustafsson and Sakai [3], the authors consider solutions of weak formulation and investigate the radius and curvature of two-dimensional moving domains respectively. For an arbitrary initial shape the moving domain its asymptote is expanding disks. Recently, progress regarding this asymptotic behavior has been made by investigating it in terms of conserved quantities, so called Richardson complex moments; see Richardson [8]. In Vondenhoff [11], by restricting multi-dimensional initial domains to be close to balls, the author gives a rescaling behavior of the moving boundaries in terms of conserved moments. In this paper, we aim to generalize the former result in two-dimensions by assuming a larger set of initial domains.

It has been known that there is a general class of polynomials which can give rise to global strong polynomial solutions to (1.1) and the corresponding initial domains can be quite different from disks; for examples, starlike polynomials (eg. $\xi + \frac{2}{5}\xi^2$ and $\frac{\xi}{1.1} - \frac{15}{14}(\frac{\xi}{1.1})^2 + \frac{4}{7}(\frac{\xi}{1.1})^3 - \frac{1}{7}(\frac{\xi}{1.1})^4$); see Gustafsson, Prokhorov and Vasil'ev [2]. An arbitrary global strong degree k_0 polynomial solution to (1.1), called $f_{k_0}(\xi, t)$, can have its rescaling behaviors precisely described in terms of moments; see Lin [5]. In this paper, as an application of Theorem 1.1, we show that small perturbation of $f_{k_0}(\xi, 0)$, called $f(\xi, 0)$, can give rise to a global strong solution $f(\xi, t)$ and a rescaling behavior of the corresponding moving domains, similar to that stated in Vondenhoff [11], is given in terms of moments as well. We can deduce the case that initial domains are small perturbation of disks from this result by letting $k_0 = 1$. Therefore, this result generalizes the result in Vondenhoff [11]. Lin [5], Vondenhoff [11] and this paper consider different sets of initial data and the rescaling behavior in Lin [5] is different from that in Vondenhoff [11] and this paper. However, geometrically, these rescaling behaviors in the three work all imply that by rescaling the corresponding moving domain $\Omega(t)$, $t \geq 0$ to be a domain $\Omega'(t)$ with area π , the radius and curvature of $\partial\Omega'(t)$ decay to 1 algebraically and the decay is faster if lower moments vanish.

The sketch of proof of this result is as the following: We first apply Theorem 1.1 and prove the existence of a locally-in-time strong solution $\{f(\xi, t)\}_{0 \leq t \leq T_0}$

where $f(\xi, T_0)$ is strongly starlike and $f(D, T_0)$ is a small perturbation of a disk, even though $f(\xi, 0)$ can be nonstarlike and $f(D, 0)$ is far from a disk. Since starlikeness is a sufficient condition for an initial function to give rise to a global strong solution as shown in Gustafsson, Prokhorov and Vasil'ev [2] and since large-time rescaling behavior for evolution of perturbed disks is shown in Vondenhoff [11] in terms of moments, the solution $f(\xi, t)$ must be global and a rescaling behavior is given in terms of moments as well.

The structure of this paper is as follows. In **Section 2**, we prove Theorem 1.1. In **Section 3**, the application of Theorem 1.1 in the suction case is given. In **Section 4**, the application of Theorem 1.1 in the injection case is given. As a byproduct of a theorem in this paper, a short proof of existence and uniqueness of strong solutions to (1.1) is given in **Section 5**.

2 Proofs of Theorem 1.1

The proof of the perturbation theorem in the suction case is almost the same as the proof in the injection case. Therefore, we will just provide the proof of the theorem in the case of injection (1.1).

As in Gustafsson [1], a reformulation of the Polubarinova-Galin equation (1.1) is expressed:

$$f_t = \xi f' P \left[\frac{1}{|f'|^2} \right], \quad \xi \in D \quad (2.1)$$

where P denotes the Poisson kernel which defines the analytic function in the unit disk

$$P[g](\xi) = \frac{1}{2\pi i} \int_{\partial D} g(z) \frac{z + \xi}{z - \xi} \frac{dz}{z}, \quad \xi \in D, \quad (2.2)$$

from boundary data g on ∂D . In the mathematical treatment of (2.1) it makes no difference if $f(\xi, t)$ is univalent in \overline{D} or merely locally univalent in \overline{D} ; see Gustafsson [1]. To make a distinction, we denote

$$\omega(E) = \left\{ f \in H(E) \mid f \text{ is locally univalent in } E, f(0) = 0 \text{ and } f'(0) > 0 \right\}$$

and define a solution to be a strong* solution to (2.1) as follows:

Definition 2.1. A solution $f(\xi, t) \in \omega(\overline{D})$ is a strong* solution to (2.1) for $0 \leq t < b$ if $f(\xi, t)$ is continuously differentiable with respect to $t \in [0, b)$ and satisfies (2.1).

An univalent strong* solution $f(\xi, t)$ to (2.1) must be a strong solution to the Polubarinova-Galin equation (1.1).

In subsection 2.1, we aim to prove a perturbation theorem for strong* polynomial solutions to (2.1), Theorem 2.4. In subsection 2.2, we show that Theorem 1.1 follows directly from Theorem 2.4.

2.1 A perturbation theorem for strong* polynomial solutions

We start with lemmas before proving the perturbation theorem for strong* polynomial solutions to (2.1).

Lemma 2.1. *For $1 < p < \infty$, there exists $C_p > 0$ such that*

$$\|P[g]\|_{L^p([0,2\pi])} \leq C_p \|g\|_{L^p([0,2\pi])}$$

for g which is holomorphic in a neighborhood of ∂D and is also a real function on ∂D .

Proof. There exists u which is harmonic in D , continuous in \overline{D} , and $u = g$ on ∂D . Therefore, by Theorem 17.26 in Rudin [9], it is shown that for $1 < p < \infty$, there exists $C_p > 0$ such that

$$\|P[u]\|_{L^p([0,2\pi])} \leq C_p \|u\|_{L^p([0,2\pi])},$$

which means

$$\|P[g]\|_{L^p([0,2\pi])} \leq C_p \|g\|_{L^p([0,2\pi])}.$$

□

In the proof of the perturbation theorem for strong* polynomial solutions, we use iterative methods. In each iteration, we need to calculate the difference of two polynomial univalent functions h_1 and h_2 which satisfy the assumption of Lemma 2.2. Inequality (2.5) enables us to estimate $\|h'_1 - h'_2\|_{L^2([0,2\pi])}$ locally in time when h_1 and h_2 are both polynomial as shown in the proof of Theorem 2.4.

Lemma 2.2. *Let $g(\xi, t) \in \omega(\overline{D_r}) \cap C^1([0, t_1], H(\overline{D_r}))$ be a strong* solution to (2.1) and $0 < l < 1$. There exists $C(g, t_1, r, l) > 0$ such that, if $h_1(z, t), h_2(z, t) \in \omega(\overline{D_r}) \cap C^1([0, t_h], H(\overline{D_r}))$ are two strong* solutions to (2.1) where $0 < t_h \leq t_1$ and*

$$\max_{([0, t_h])} |h'_i(\cdot, t) - g'(\cdot, t)|_{M(r)} \leq l \min_{(\overline{D_r}, [0, t_1])} |g'|, \quad 1 \leq i \leq 2, \quad (2.3)$$

then we have

$$\left\| \frac{\partial}{\partial t} [h_1 - h_2] \right\|_{L^2([0, 2\pi])} \leq C \|h'_1 - h'_2\|_{L^2([0, 2\pi])}, \quad 0 \leq t \leq t_h. \quad (2.4)$$

Furthermore, if h_1, h_2 are both polynomials of degree $\leq n$, then for $0 \leq t \leq t_h$

$$\left\| h'_1(\cdot, t) - h'_2(\cdot, t) \right\|_{L^2([0, 2\pi])}^2 \leq e^{2C(n)t} \left\| h'_1(\cdot, 0) - h'_2(\cdot, 0) \right\|_{L^2([0, 2\pi])}^2. \quad (2.5)$$

Proof. (1)

$$\frac{\partial}{\partial t} [h_1 - h_2] = \xi \left\{ [h'_1 - h'_2] P \left[\frac{1}{|h'_2|^2} \right] + h'_1 P \left[\frac{1}{|h'_1|^2} - \frac{1}{|h'_2|^2} \right] \right\}. \quad (2.6)$$

Here, by Lemma 2.1,

$$\left\| P \left[\frac{1}{|h_1'|^2} - \frac{1}{|h_2'|^2} \right] \right\|_{L^2([0, 2\pi])} \leq C_2 \left\| \frac{1}{|h_1'|^2} - \frac{1}{|h_2'|^2} \right\|_{L^2([0, 2\pi])}. \quad (2.7)$$

By taking the L_2 norm of the right-hand side and the left-hand side of (2.6) and then using (2.7) and Hölder's inequality, we obtain

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} [h_1 - h_2] \right\|_{L^2([0, 2\pi])} \\ & \leq \|h_1' - h_2'\|_{L^2([0, 2\pi])} \max_{\partial D} \left| P \left[\frac{1}{|h_2'|^2} \right] \right| + C_2 \|h_1'\|_{L^\infty([0, 2\pi])} \left\| \frac{1}{|h_1'|^2} - \frac{1}{|h_2'|^2} \right\|_{L^2([0, 2\pi])} \\ & \leq \left\{ \max_{\partial D} \left| P \left[\frac{1}{|h_2'|^2} \right] \right| + C_2 \max_{\partial D} |h_1'| \max_{\partial D} \frac{|h_1'| + |h_2'|}{|h_1'|^2 |h_2'|^2} \right\} \|h_1' - h_2'\|_{L^2([0, 2\pi])} \end{aligned} \quad (2.8)$$

We want to bound

$$\max_{\partial D} |h_1'| \max_{\partial D} \frac{|h_1'| + |h_2'|}{|h_1'|^2 |h_2'|^2} \quad \text{and} \quad \max_{\partial D} \left| P \left[\frac{1}{|h_2'|^2} \right] \right|$$

respectively in (i) and (ii) in terms of g and hereby determine the constant C .

(i) By assumption (2.3), for $(z, t) \in (\partial D, [0, t_h])$,

$$|h_i'(z, t)| \geq |g'(z, t)| - |h_i'(z, t) - g'(z, t)| \geq (1 - l) |g'(z, t)|, \quad 1 \leq i \leq 2 \quad (2.9)$$

and

$$|h_i'(z, t)| \leq |g'(z, t)| + |h_i'(z, t) - g'(z, t)| \leq (1 + l) |g'(z, t)|, \quad 1 \leq i \leq 2. \quad (2.10)$$

Therefore, by (2.9) and (2.10), for $0 \leq t \leq t_h$

$$\max_{\partial D} |h_1'| \max_{\partial D} \frac{|h_1'| + |h_2'|}{|h_1'|^2 |h_2'|^2} \leq 2 \frac{(1 + l)}{(1 - l)^3} \max_{(\partial D, [0, t_1])} |g'| \max_{(\partial D, [0, t_1])} \frac{1}{|g'|^3}. \quad (2.11)$$

(ii) We start with finding the upper bound of $P \left[\frac{1}{|h_2'|^2} - \frac{1}{|g'|^2} \right]$ in terms of g and

hereby obtain the upper bound for $P \left[\frac{1}{|h_2'|^2} \right]$ in terms of g .

In Gustafsson [1], it is shown that for given $h \in \omega(\overline{D_r})$,

$$P \left[\frac{1}{|h'|^2} \right] = \frac{1}{2\pi i} \int_{\partial D_r} \frac{1}{h'(z, t) \overline{h'(1/z, t)}} \frac{z + \xi}{z - \xi} \frac{dz}{z}, \quad \xi \in D. \quad (2.12)$$

By (2.12), we have for $\xi \in D$

$$P \left[\frac{1}{|h'_2|^2} - \frac{1}{|g'|^2} \right] = \frac{1}{2\pi i} \int_{\partial D_r} \left(\frac{1}{h'_2(z, t) \overline{h'_2}(1/z, t)} - \frac{1}{g'(z, t) \overline{g'}(1/z, t)} \right) \frac{z + \xi}{z - \xi} \frac{dz}{z}. \quad (2.13)$$

Therefore,

$$\begin{aligned} & \max_{\partial D} \left| P \left[\frac{1}{|h'_2|^2} - \frac{1}{|g'|^2} \right] \right| \\ & \leq \max_{\partial D_r} \left| \frac{1}{h'_2(z, t) \overline{h'_2}(1/z, t)} - \frac{1}{g'(z, t) \overline{g'}(1/z, t)} \right| \frac{r+1}{r-1} \\ & = \max_{\partial D_r} \left| \frac{h'_2(z, t) - g'(z, t)}{g'(z, t) \overline{g'}(1/z, t) h'_2(z, t)} + \frac{\overline{h'_2}(1/z, t) - \overline{g'}(1/z, t)}{h'_2(z, t) \overline{h'_2}(1/z, t) \overline{g'}(1/z, t)} \right| \frac{r+1}{r-1}. \end{aligned} \quad (2.14)$$

By assumption (2.3), for $(z, t) \in (\partial D_r, [0, t_h])$,

$$\left| \overline{h'_2}(1/z, t) - \overline{g'}(1/z, t) \right| \leq l \left| \overline{g'}(1/z, t) \right|, \quad (2.15)$$

$$\left| \overline{h'_2}(1/z, t) \right| \geq (1-l) \left| \overline{g'}(1/z, t) \right|. \quad (2.16)$$

By assumption (2.3), for $(z, t) \in (\partial D_r, [0, t_h])$,

$$\left| h'_2(z, t) - g'(z, t) \right| \leq l \left| g'(z, t) \right|, \quad (2.17)$$

$$\left| h'_2(z, t) \right| \geq (1-l) \left| g'(z, t) \right|. \quad (2.18)$$

By (2.15)-(2.18),

$$\begin{aligned} & \max_{(\partial D_r, [0, t_h])} \left| \frac{h'_2(z, t) - g'(z, t)}{g'(z, t) \overline{g'}(1/z, t) h'_2(z, t)} + \frac{\overline{h'_2}(1/z, t) - \overline{g'}(1/z, t)}{h'_2(z, t) \overline{h'_2}(1/z, t) \overline{g'}(1/z, t)} \right| \frac{r+1}{r-1} \\ & \leq 2l \left[\max_{(\partial D_r, [0, t_1])} \left| \frac{1}{|g'(z, t)| \left| \overline{g'}(1/z, t) \right| (1-l)^2} \right| \right] \frac{r+1}{r-1}. \end{aligned}$$

Therefore, by the above inequality and (2.14), for $0 \leq t \leq t_h$

$$\begin{aligned} & \max_{\partial D} \left| \xi P \left[\frac{1}{|h'_2|^2} - \frac{1}{|g'|^2} \right] \right| \\ & \leq 2l \left[\max_{(\partial D_r, [0, t_1])} \left| \frac{1}{|g'(z, t)| \left| \overline{g'}(1/z, t) \right| (1-l)^2} \right| \right] \frac{r+1}{r-1}. \end{aligned} \quad (2.19)$$

Hence, for $0 \leq t \leq t_h$, we have

$$\begin{aligned}
& \max_{\partial D} \left| \xi P \left[\frac{1}{|h_2'|^2} \right] \right| \\
& \leq \max_{\partial D} \left| \xi P \left[\frac{1}{|h_2'|^2} - \frac{1}{|g'|^2} \right] \right| + \max_{\partial D} \left| \xi P \left[\frac{1}{|g'|^2} \right] \right| \\
& \leq 2l \left[\max_{(\partial D_r, [0, t_1])} \left| \frac{1}{|g'(z, t)| |\overline{g'}(1/z, t)| (1-l)^2} \right| \right] \frac{r+1}{r-1} + \max_{(\partial D, [0, t_1])} \left| \xi P \left[\frac{1}{|g'|^2} \right] \right|.
\end{aligned} \tag{2.20}$$

From (i) and (ii), we prove (2.4) by choosing C to be

$$\begin{aligned}
C &= \max_{(\partial D, [0, t_1])} \left| \xi P \left[\frac{1}{|g'|^2} \right] \right| + 2l \left[\max_{(\partial D_r, [0, t_1])} \left| \frac{1}{|g'(z, t)| |\overline{g'}(1/z, t)| (1-l)^2} \right| \right] \frac{r+1}{r-1} \\
&+ C_2 \left\{ 2 \frac{(1+l)}{(1-l)^3} \max_{(\partial D, [0, t_1])} |g'| \max_{(\partial D, [0, t_1])} \frac{1}{|g'|^3} \right\}.
\end{aligned} \tag{2.21}$$

(2) Now we assume that h_1, h_2 are both polynomials of degree $\leq n$. Denote $h_1 = \sum_{i=1}^n \alpha_i(t) \xi^i$ and $h_2 = \sum_{i=1}^n \beta_i(t) \xi^i$. Also denote $D(t)$ by

$$D(t) = \left\| h_1' - h_2' \right\|_{L^2([0, 2\pi])}^2 = 2\pi \left\{ \left(\sum_{i=1}^n [|\alpha_i(t) - \beta_i(t)|^2 i^2] \right) \right\}.$$

Then

$$\begin{aligned}
D'(t) &= 2\pi \cdot 2 \left\{ \left(\sum_{i=1}^n \operatorname{Re} \left[(\alpha_i - \beta_i) \overline{(\alpha_i - \beta_i)_t} \right] i^2 \right) \right\} \\
&\leq 2\pi \cdot 2(n) \left\{ \left(\sum_{i=1}^n |(\alpha_i - \beta_i)| |(\alpha_i - \beta_i)_t| i \right) \right\} \\
&\leq 2\pi \cdot 2(n) \left\{ \left(\sum_{i=1}^n |(\alpha_i - \beta_i)|^2 i^2 \right) \right\}^{\frac{1}{2}} \left\{ \left(\sum_{i=1}^n |(\alpha_i - \beta_i)_t|^2 \right) \right\}^{\frac{1}{2}} \\
&= 2(n) \left\| [h_1' - h_2'] \right\|_{L^2([0, 2\pi])} \left\| \frac{\partial}{\partial t} [h_1 - h_2] \right\|_{L^2([0, 2\pi])}.
\end{aligned}$$

By applying (2.4) to the above inequality, we conclude that for $0 \leq t \leq t_h$,

$$D'(t) \leq 2C(n) \left\| [h_1' - h_2'] \right\|_{L^2([0, 2\pi])}^2 = 2C(n) D(t), \tag{2.22}$$

and therefore

$$D(t) \leq D(0) e^{2Ct(n)}, \tag{2.23}$$

which proves (2.5). \square

The following lemma helps us to control the blow-up time of strong* polynomial solutions to (2.1).

Lemma 2.3. *Given a polynomial mapping $f(\xi, 0) \in \omega(\overline{D_{r_0}})$ for some $r_0 > 1$, then there exists a unique strong* polynomial solution to (2.1) $f(\xi, t) \in \omega(\overline{D_{r_0}})$ at least for a short time. Furthermore, if the strong* polynomial solution ceases to exist at $t = b$, then for any $r > 1$,*

$$\liminf_{t \rightarrow b} \left(\min_{\overline{D_r}} |f'(\xi, t)| \right) = 0. \quad (2.24)$$

Proof. (a) The first part follows from Gustafsson [1].

(b) Assume that (2.24) does not hold now. Then there exists $r > 1$ such that

$$\liminf_{t \rightarrow b} \left(\min_{\overline{D_r}} |f'(\xi, t)| \right) > 0.$$

This implies that there exist $C > 0$ and $1 < r' \leq r$ such that

$$\min_{\overline{D_{r'}}} |f'(\xi, t)| > C, \quad t \in [0, b).$$

Since each coefficient of $f(\xi, t)$ is bounded for $t \in [0, b)$, there exists $M > 0$ such that

$$\sup_{t \in [0, b)} \max_{\overline{D_{r'}}} |f'(\xi, t)\xi| \leq M.$$

For $\xi \in \overline{D}$

$$\begin{aligned} & \sup_{t \in [0, b)} \left| f'(\xi, t) \xi P \left[\frac{1}{|f'|^2} \right] \right| \\ & \leq \sup_{t \in [0, b)} \left| \frac{f'(\xi, t)\xi}{2\pi i} \int_{\partial D_{r'}} \frac{1}{f'(z, t)\overline{f'(1/z, t)}} \frac{z + \xi}{z - \xi} \frac{dz}{z} \right| \\ & \leq \sup_{t \in [0, b)} \left(\max_{\overline{D}} |f'(\xi, t)\xi| \cdot \max_{\partial D_{r'}} \left| \frac{1}{f'(z, t)\overline{f'(1/z, t)}} \right| \frac{r' + 1}{r' - 1} \right) \\ & \leq \frac{M}{C^2} \frac{r' + 1}{r' - 1} \end{aligned}$$

Therefore, for $0 \leq t_2 < t_1 < b$, $\xi \in D$

$$|f(\xi, t_1) - f(\xi, t_2)| = \left| \int_{t_2}^{t_1} f'(\xi, t) \xi P \left[\frac{1}{|f'|^2} \right] dt \right| \leq |t_1 - t_2| \frac{M}{C^2} \frac{r' + 1}{r' - 1}.$$

Therefore $\lim_{t \rightarrow b} f(\xi, t)$ exists and we define it as $f(\xi, b)$. Note that $f(\xi, b)$ satisfies $\min_{\overline{D_{r'}}} |f'(\xi, b)| \geq C$. Let $f(\xi, t + b)$ be the strong* solution to (2.1) with

the initial value $f(\xi, b)$ for $t \in [0, \epsilon)$. Then $f(\xi, t)$ is continuous with respect to t for $t \in [0, b + \epsilon)$ and

$$f(\xi, t) - f(\xi, 0) = \int_0^t f'(\xi, s) \xi P \left[\frac{1}{|f'(\cdot, s)|^2} \right] ds.$$

This implies that $f(\xi, t) \in \omega(\overline{D})$ is continuously differentiable with respect to t for $t \in [0, b + \epsilon)$ and satisfies (2.1). Hence it is impossible that $f(\xi, t)$ ceases to exist at $t = b$ and therefore for any $r > 1$,

$$\liminf_{t \rightarrow b} \left(\min_{\overline{D}_r} |f'(\xi, t)| \right) = 0.$$

□

Theorem 2.4. Assume that $f_{k_0}(\xi, t) \in C^1([0, t_1], H(\overline{D_r})) \cap \omega(\overline{D_r})$ is a strong* degree k_0 polynomial solution to (2.1) for some $t_1 > 0$ and $r > 1$ and that $\rho > r$ and $l < 1$. If $f(\xi, 0)$ satisfies the assumption (A)

$$\|f(\xi, 0) - f_{k_0}(\xi, 0)\|_{\rho, 1} \leq \frac{l}{\sqrt{k_0}} \min_{(\overline{D_r}, [0, t_1])} |f'_{k_0}|$$

where $f'(0, 0) \in R$ and $f(0, 0) = 0$, then the following (a)-(b) are true:

(a) There exists $C(f_{k_0}, t_1, r, l) > 0$ such that a strong* solution to (2.1) $f(\xi, t) \in C^1([0, t_0], H(D_r) \cap C(\overline{D_r})) \cap \omega(D_r)$ where $t_0 = \min \left\{ \frac{1}{C_{k_0}} (\ln \frac{\rho}{r}), t_1 \right\}$. Moreover,

$$\max_{([0, t_0])} |f' - f'_{k_0}|_{M(r)} \leq l \min_{(\overline{D_r}, [0, t_1])} |f'_{k_0}|.$$

(b) Furthermore, if there exist $\delta > 0$ and j nonnegative integer such that

$$\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\rho, j} \leq \delta,$$

then there exists $c(j, k_0) > 0$ such that

$$\max_{([0, t_0])} |f^{(j)} - f_{k_0}^{(j)}|_{M(r)} \leq c(j, k_0) \delta.$$

Remark 2.2. The strong* solution $f(\xi, t)$ is obtained by using many polynomial strong* solutions to (2.1) to approximate it.

Proof. (a) We take the constant $C(f_{k_0}, t_1, r, l)$ in (a) to be the same as the one defined in Lemma 2.2. We want to prove (a) in the following, by showing that there exists a strong* solution $f(\xi, t) \in \omega(D_r)$ to (2.1) for $0 \leq t \leq t_0$, where $f(\xi, 0) = f_{k_0}(\xi, 0) + \sum_{i=1}^{\infty} b_i(0) \xi^i$ and

$$\sum_{k=1}^{\infty} |b_k(0)| \rho^k k^{3/2} \leq \frac{l}{\sqrt{k_0}} \min_{(\overline{D_r}, [0, t_1])} |f'_{k_0}|. \quad (2.25)$$

Denote the strong* polynomial solution to (2.1) with the initial value $f_{k_0}(\xi, 0) + \sum_{i=1}^k b_i(0)\xi^i$ by $g_k(\xi, t)$. The proof for (a) is split into step1 and step2. In step1, we prove that $g_k(\xi, t), k \geq 1$ exists for $t \in [0, t_0]$. In step2, we prove that $g_k(\xi, t)$ converges to the strong* solution $f(\xi, t)$ as k goes to infinity and that $f(\xi, t)$ exists for $t \in [0, t_0]$.

Step1:

By (2.25), there exist $\{d_k\}_{k \geq 0}$ nonnegative and $\sum_{k=0}^{\infty} d_k = 1$ such that $|b_i(0)| \leq M_i \rho^{-i}$ for $i \geq 1$ where

$$M_{k+1} \leq \frac{l}{\sqrt{k_0}} \frac{d_k}{(k+1)^{3/2}} \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |f'_{k_0}|, \quad k \geq 0.$$

Claim:

Prove that for $k \geq 0$, $g_k(\xi, t) \in C^1([0, t_0], H(\overline{\mathcal{D}_r})) \cap \omega(\overline{\mathcal{D}_r})$ and

$$\max_{([0, t_0])} |g'_k - g'_{k+1}|_{M(r)} \leq l d_k \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0|.$$

Proof. (proof of claim) We prove it by induction as follows.

(i) Assume for $0 \leq k \leq n-1$,

$$\max_{([0, t_0])} |g'_k - g'_{k+1}|_{M(r)} \leq l d_k \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0|.$$

(ii) **Subclaim:**

For $t \in [0, t_0]$

$$|g'_n - g'_{n+1}|_{M(r)} \leq l d_n \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0|. \quad (2.26)$$

Proof. (of subclaim) Denote $s_n = \sup\{T \leq t_0 | g_{n+1}(\xi, t) \text{ satisfies (2.26) for } t \in [0, T]\}$.

Then $|g'_{n+1}| \geq (1-l)|g'_0|$ for $t \in [0, s_n]$. Therefore, by Lemma 2.3, the value $s_n = \max\{T \leq t_0 | g_{n+1}(\xi, t) \text{ satisfies (2.26) for } t \in [0, T]\}$.

For $0 < t \leq s_n$,

$$\max_{([0, t])} |g'_{n+1} - g'_0|_{M(r)} \leq \sum_{k=0}^n l d_k \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0| \leq l \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0|. \quad (2.27)$$

Also by the assumption in (i), we have

$$\max_{([0, t_0])} |g'_n - g'_0|_{M(r)} \leq \sum_{k=0}^{n-1} l d_k \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0| \leq l \min_{(\overline{\mathcal{D}_r}, [0, t_1])} |g'_0|. \quad (2.28)$$

From (2.27) and (2.28), g_0, g_n and g_{n+1} satisfy the assumption for g, h_1 and h_2 in Lemma 2.2 respectively. Denote $D(t) = \|g'_{n+1} - g'_n\|_{L^2([0, 2\pi])}^2$. From Lemma 2.2, we can obtain that for $0 \leq t \leq s_n$

$$D(t) \leq e^{2C(n+1)k_0 t} D(0). \quad (2.29)$$

We need to show $s_n = t_0$.

Note that if $s_n < t_0$, then the following (R_1) must hold:

(R_1) At time $t = s_n$,

$$\left| g_n' - g_{n+1}' \right|_{M(r)} = d_n \min_{(\overline{D_r}, [0, t_1])} \left| g_0' \right| l.$$

Assume that $s_n < t_0$ now. Then for $0 \leq t \leq s_n$,

$$\begin{aligned} & \left| g_n' - g_{n+1}' \right|_{M(r)} \\ & \leq \sqrt{D(t)(n+1)k_0 r^{(n)}} \\ & \leq \sqrt{(n+1)k_0 D(0) e^{2Ct k_0(n+1)} r^{2(n)}} \\ & \leq \sqrt{(n+1)k_0 D(0) e^{2Cs_n k_0(n+1)} r^{2(n)}} \\ & < \sqrt{(n+1)k_0 D(0) e^{2Ct_0 k_0(n+1)} r^{2(n)}}. \end{aligned}$$

Since

$$D(0)(n+1)k_0 \leq (\rho)^{-2(n+1)} (d_n)^2 \min_{(\overline{D_r}, [0, t_1])} \left| g_0' \right|^2 l^2,$$

we have

$$\begin{aligned} & \max_{([0, s_n])} \left| g_n' - g_{n+1}' \right|_{M(r)} \\ & \leq \sqrt{(n+1)k_0 D(0) e^{2Ct_0 k_0(n+1)} r^{2(n)}} \\ & < d_n \min_{(\overline{D_r}, [0, t_1])} \left| g_0' \right| l \end{aligned}$$

which contradicts the remark (R_1) . Therefore, $s_n = t_0$. □

□

Step2:

By Step 1, for $k \geq 1$

$$\max_{([0, t_0])} \left| g_k' - g_0' \right|_{M(r)} \leq l \sum_{n=0}^{\infty} d_n \min_{(\overline{D_r}, [0, t_1])} \left| g_0' \right| \leq l \min_{(\overline{D_r}, [0, t_1])} \left| g_0' \right|.$$

There exists $f(\xi, t) \in C([0, t_0], \omega(D_r) \cap C(\overline{D_r}))$ such that $|g_k' - f'|_{M(r)}$ goes to zero as k goes to ∞ . Furthermore,

$$\max_{([0, t_0])} \left| f' - g_0' \right|_{M(r)} \leq l \min_{(\overline{D_r}, [0, t_1])} \left| g_0' \right|.$$

Still, we have to show that $f(\xi, t)$ satisfies (2.1). Fix $1 < r' < r$. For $\xi \in D_{r'}$ and $0 \leq t \leq t_0$,

$$\frac{\partial}{\partial t} g_k(\xi, t) = \frac{g_k'(\xi, t)\xi}{2\pi i} \int_{\partial D_{r'}} \frac{1}{g_k'(z, t) \overline{g_k'(1/z, t)}} \frac{z + \xi}{z - \xi} \frac{dz}{z}. \quad (2.30)$$

By integrating (2.30) with respect to t , we have that for $\xi \in D_{r'}$ and $0 \leq t \leq t_0$,

$$g_k(\xi, t) - g_k(\xi, 0) = \int_0^t \frac{g'_k(\xi, s)\xi}{2\pi i} \int_{\partial D_{r'}} \frac{1}{g'_k(z, s)\overline{g'_k(1/z, s)}} \frac{z + \xi}{z - \xi} \frac{dz}{z} ds.$$

Let $k \rightarrow \infty$. For ξ in any compact subset of $D_{r'}$,

$$f(\xi, t) - f(\xi, 0) = \int_0^t \frac{f'(\xi, s)\xi}{2\pi i} \int_{\partial D_{r'}} \frac{1}{f'(z, s)\overline{f'(1/z, s)}} \frac{z + \xi}{z - \xi} \frac{dz}{z} ds \quad (2.31)$$

for some $f(\xi, t) \in C([0, t_0], \omega(D_r) \cap C(\overline{D_r}))$. Furthermore, the identity (2.31) shows that $f(\xi, t) \in C^1([0, t_0], H(D_r) \cap C(\overline{D_r}))$.

(b) Now assume (b). Then

$$|b_i(0)| \leq M_i \rho^{-i}, \quad i \geq 1$$

where

$$M_{k+1} \leq \frac{1}{(k+1)^{\frac{1}{2}+j}} d_k \delta, \quad k \geq 0.$$

First we look at the case $j = 2$. Under (b),

$$\begin{aligned} & \max_{([0, t_0])} |g''_n - g''_{n+1}|_{M(r)} \\ & \leq \sqrt{(n+2)^3(k_0+1)^3 \frac{1}{3} D(0) e^{2Ct_0 k_0(n+1)} r^{n-1}} \\ & = \left(\frac{n+2}{n+1}\right)^{\frac{3}{2}} \frac{1}{\sqrt{3}} (k_0+1)^{\frac{3}{2}} \sqrt{D(0)(n+1)^3 e^{2Ct_0 k_0(n+1)} r^{n-1}} \\ & \leq \left(\frac{n+2}{n+1}\right)^{\frac{3}{2}} \frac{1}{\sqrt{3}} (k_0+1)^{\frac{3}{2}} d_n \delta, \quad n \geq 0. \end{aligned}$$

Therefore, we have for $n \geq 1$

$$\max_{([0, t_0])} |g''_0 - g''_n|_{M(r)} \leq \frac{1}{\sqrt{3}} 2^{\frac{3}{2}} (k_0+1)^{\frac{3}{2}} \delta.$$

Similarly, for $j \geq 2$, under the assumption of (b), there exists $c(j, k_0) > 0$ such that

$$\begin{aligned} & \max_{([0, t_0])} |g_n^{(j)} - g_{n+1}^{(j)}|_{M(r)} \\ & \leq c(j, k_0) \sqrt{(n+1)^{2j-1} D(0) e^{2Ct_0 k_0(n+1)}} \\ & \leq c(j, k_0) d_n \delta. \end{aligned}$$

Therefore, we have

$$\max_{([0, t_0])} |g_0^{(j)} - g_n^{(j)}|_{M(r)} \leq c(j, k_0) \delta.$$

Let $n \rightarrow \infty$,

$$\max_{([0, t_0])} |g_0^{(j)} - f^{(j)}|_{M(r)} \leq c(j, k_0) \delta.$$

□

2.2 A perturbation theorem for strong polynomial solutions

In the former subsection, the solutions we considered are locally univalent in \overline{D} . However, the solutions which have physical meaning are required to be univalent in \overline{D} . The following Lemma 2.5 states that these locally univalent solutions are univalent if they are close to a univalent solution.

Lemma 2.5. *Given $g(\xi, t) \in C^1([0, T_0], H(\overline{D_r})) \cap O(\overline{D_r})$ and $1 < r' < r$, there exists $\eta(g, T_0, r') > 0$ such that if*

$$\max_{([0, T_0])} \left| f'(\cdot, t) - g'(\cdot, t) \right|_{M(r)} \leq \eta$$

where $f(\xi, t) \in C([0, T_0], H(D_r) \cap C(\overline{D_r}))$, then for $0 \leq t \leq T_0$,

$$f(\xi, t) \in O(\overline{D_{r'}}).$$

Proof. The proof is separated into two parts (a)-(b):

(a) First assume that

$$\max_{([0, T_0])} \left| f'(\cdot, t) - g'(\cdot, t) \right|_{M(r)} \leq \frac{1}{2} \min_{(\overline{D_r}, [0, T_0])} \left| g'(z, t) \right|. \quad (2.32)$$

We want to show that there exists $r_0 > 0$ such that for any fixed $z_0 \in \overline{D_{r'}}$,

$$f(\cdot, t) : \overline{D_{r_0}(z_0)} \rightarrow f(\overline{D_{r_0}(z_0)})$$

is univalent. It is sufficient to prove that

$$\operatorname{Re} \frac{f'(z, t)(z - z_0)}{f(z, t) - f(z_0, t)} \geq \frac{1}{2}, \quad z \in D_{r_0}(z_0)$$

which means the function is injective on $\partial D_{r_0}(z_0)$ and therefore is injective for $z \in \overline{D_{r_0}(z_0)}$.

Now fix $z_0 \in \overline{D_{r'}}$. Since $f(z, t)$ is analytic in D_r ,

$$f(z, t) = f(z_0, t) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0, t)}{n!} (z - z_0)^n, \quad z \in D_r.$$

Let

$$l = \min\{r', r - r'\}, M = \frac{3}{2} \max_{(\overline{D_r}, [0, T_0])} |g'|, m = \frac{1}{2} \min_{(\overline{D_r}, [0, T_0])} |g'|.$$

By (2.32), we can get that

$$\max_{(\overline{D_r}, [0, T_0])} |f(z, t)| \leq M, \quad \min_{(\overline{D_r}, [0, T_0])} |f'(z, t)| \geq m.$$

Note that

$$\left| \frac{f^{(n)}(z_0, t)}{n!} \right| \leq M l^{-(n)}, \quad n \geq 1.$$

Pick $0 < r_0 < l$ such that $\sum_{n=2}^{\infty} Ml^{-n}r_0^{n-1}(n-1) \leq \frac{m}{4}$. For $|z - z_0| < r_0$, we have

$$\begin{aligned} & \left| \frac{f'(z, t)(z - z_0)}{f(z, t) - f(z_0, t)} - 1 \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0, t)}{n!} (z - z_0)^{n-1} n}{\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0, t)}{n!} (z - z_0)^{n-1}} - 1 \right| \\ &= \left| \frac{\sum_{n=2}^{\infty} \frac{f^{(n)}(z_0, t)}{n!} (z - z_0)^{n-1} (n-1)}{f'(z_0, t) + \sum_{n=2}^{\infty} \frac{f^{(n)}(z_0, t)}{n!} (z - z_0)^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} Ml^{-n} |z - z_0|^{n-1} (n-1)}{m - \sum_{n=2}^{\infty} Ml^{-n} |z - z_0|^{n-1}} \leq \frac{1}{2}. \end{aligned}$$

It follows from the above inequality that

$$Re \frac{f'(z, t)(z - z_0)}{f(z, t) - f(z_0, t)} \geq \frac{1}{2}, \quad z \in D_{r_0}(z_0).$$

(b) Assume that there doesn't exist such $\eta > 0$ such that the Lemma holds, then there exist $\eta_k, f^k(\xi, t) \in C^1([0, T_0], H(D_r) \cap C(\overline{D_r}))$ and $\xi_k^1, \xi_k^2 \in \overline{D_{r'}}$ where $\xi_k^1 \neq \xi_k^2$, such that

- (1) η_k goes to zero as k goes to ∞ ;
- (2) $f^k(\xi_k^1, t_k) = f^k(\xi_k^2, t_k)$;
- (3) $|f^k(\xi_k^1, t_k) - g(\xi_k^1, t_k)| \leq \eta_k, |f^k(\xi_k^2, t_k) - g(\xi_k^2, t_k)| \leq \eta_k$.

Without loss of generality, assume t_k converges to t_0 , ξ_k^1 converges to ξ^1 and ξ_k^2 converges to ξ^2 . Note that $|\xi^1 - \xi^2| \geq r_0$. This implies

$$g(\xi^1, t_0) = g(\xi^2, t_0).$$

This contradicts the assumption that $g(\xi, t_0)$ is univalent in $\overline{D_r}$. Therefore, there exists $\eta > 0$ such that the Lemma holds. \square

Proof. (proof of Theorem 1.1)

(a) By Lemma 2.5, there exists $\eta(f_{k_0}, T_0, r') > 0$ such that if $f(\xi, t)$ satisfies

$$f(\xi, t) \in C^1([0, T_0], H(D_r)) \quad \text{and} \quad \max_{([0, T_0])} |f'_{k_0}(\cdot, t) - f'(\cdot, t)|_{M(r)} \leq \eta,$$

then $f(\xi, t) \in O(\overline{D_{r'}})$ for $t \in [0, T_0]$.

(b) We apply Theorem 2.4 by letting $t_1 = T_0$, $l = \frac{1}{2}$, δ small enough such that

$$\delta < \min_{1 \leq j \leq k} \left\{ \frac{\epsilon}{c(j, k_0)} \right\}, \quad \delta < \frac{l}{\sqrt{k_0}} \min_{(\overline{D_r}, [0, T_0])} |f'_{k_0}(\xi, t)|, \quad \delta < \min_{1 \leq j \leq k} \left\{ \frac{\eta}{c(j, k_0)} \right\}$$

and $\rho > 1$ large enough such that $\frac{1}{C_{k_0}}(\ln \rho - \ln r) \geq T_0$. We get that for $0 \leq n \leq k$, $0 \leq t \leq T_0$, the strong* solution to (2.1) $f(\xi, t)$ satisfies

$$|f_{k_0}^{(n)}(\cdot, t) - f^{(n)}(\cdot, t)|_{M(r)} < \min\{\epsilon, \eta\}.$$

Therefore $f(\xi, t) \in O(\overline{D_{r'}})$ and hence is a strong solution to (1.1). \square

3 Application-Evolution of perturbed disks in the suction case

In this section, we aim to characterize the evolution of perturbed disks in the suction case.

Lemma 3.1. *Given $f_{k_0}(\xi, 0) \in O(\overline{D})$ which is a polynomial of degree k_0 . Let $f_{k_0}(\xi, t)$ be the strong solution to (1.2) and the strong solution cease to exist as $t = b$. Then given $0 < T_0 < b$, there exist $\rho > 1$ and $\delta > 0$ such that, if $\|f(\xi, 0) - f_{k_0}(\xi, 0)\|_{\rho, 1} < \delta$, then the solution $f(\xi, t)$ to (1.2) exists for $0 \leq t \leq T_0$.*

Proof. (a) There exists $r > 1$ such that $f_{k_0}(\xi, t) \in O(\overline{D_r})$ for all $0 < t < T_0$.

(b) By Theorem 1.1, we are done with the proof. \square

Theorem 3.2. *If the initial domain is close to a disk, then most of fluid is sucked before the corresponding strong solution to (1.2) blows up.*

Proof. Assume that the disk is with area π and therefore the conformal mapping is $f_1(\xi, 0) = \xi$. The strong solution to (1.2) is $f_1(\xi, t) = \sqrt{1 - 2t}\xi$ and the fluid is sucked out as $t = b = \frac{1}{2}$.

For $T_0 < b$, we apply Lemma 3.1 and obtain that there exist $\rho > 1$ and $\delta > 0$ such that, if $\|f(\xi, 0) - f_1(\xi, 0)\|_{\rho, 1} < \delta$, then the solution $f(\xi, t)$ to (1.2) exists for $0 < t < T_0$. If $b - T_0$ is small, the results show that most of fluid will be sucked before the strong solution $f(\xi, t)$ blows up. \square

4 Application-Large-time rescaling behaviors for large data and moments in the injection case

In Richardson [8], given $\Omega(t)$ which solves the Hele-Shaw problem with injection, the Richardson complex moments $\{M_k(t)\}_{k \geq 0}$ are defined by

$$M_k(t) = \frac{1}{\pi} \int_{\Omega(t)} z^k dx dy, \quad z = x + iy.$$

The quantity $M_0(t)\pi = \sqrt{2t + M_0(0)}\pi$ is the area of $\Omega(t)$ and $M_k(t), k \geq 1$ are conserved. Denote $\Omega'(t) = \left\{ \frac{x}{\sqrt{2t + M_0(0)}} \mid x \in \Omega(t) \right\}$ which has area π always.

Recall the definition of a strongly starlike function as in Gustafsson, Prokhorov and Vasil'ev [2] and Pommerenke [6]. A function $f \in O(D)$ is said to be strongly starlike if there exists $\alpha \in (0, 1]$ such that

$$\left| \arg \frac{\xi f'(\xi)}{f(\xi)} \right| < \alpha \frac{\pi}{2}, \quad \xi \in D.$$

Such a function is also called a strongly starlike function of order α .

In the case that $\Omega(t) = f(D, t)$ where $f(\xi, t)$ is a global strong solution and is strongly starlike for $t \geq T_0$, $\partial\Omega'(t), t \geq T_0$ can be expressed by a polar coordinate

equation $(1 + \bar{r}_f(t, \theta), \theta)$ for some $\bar{r}_f(t, \cdot) : S^1 \rightarrow [-1, \infty)$. The function $\bar{r}_f(t, \theta)$ satisfies

$$\bar{r}_f(t, \theta) = \frac{|f(\xi, t)|}{\sqrt{2t + M_0(0)}} - 1, \quad t \geq T_0$$

where $\theta = \arg \frac{f(\xi, t)}{|f(\xi, t)|}$ for ξ on ∂D . The value $\bar{r}_f(t, \theta)$ is well-defined if the function $f(\xi, t)$ is strongly starlike.

Define $M_k(f)$, $k \geq 1$ to be the moments corresponding to the moving domain $\Omega(t) = f(D, t)$ where $f(\xi, t)$ is a strong solution to (1.1). In this section, we aim to prove Theorem 4.1 as follows:

Theorem 4.1. *Given a global strong degree k_0 polynomial solution to (1.1) $\{f_{k_0}(\xi, t)\}_{t \geq 0}$.*

(a) *There exist $\rho(f_{k_0}) > 1, \delta(f_{k_0}) > 0, T_0(f_{k_0}) > 0$ such that if $\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\rho, 3} < \delta$, then the strong solution to (1.1) $f(\xi, t)$ is global and is a family of strongly starlike functions of order < 1 for $t \geq T_0$.*

(b) *If $n_0 = \min\{k \geq 1 \mid M_k(f) \neq 0\}$, then*

$$\lim_{T_0 \leq t \rightarrow \infty} \|\bar{r}_f(t, \cdot)\|_{C^{2, \alpha}(S^1)}(t)^\lambda = 0, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right),$$

where $\bar{r}_f(t, \theta) = \frac{|f(\xi, t)|}{\sqrt{2t + M_0(0)}} - 1$ and $\theta = \arg f(\xi, t)$, which are well-defined for $t \geq T_0$.

The proof of Theorem 4.1 is given in subsection 4.1. A geometric characterization of results in Theorem 4.1 is given in subsection 4.2.

4.1 Proofs for Theorem 4.1

We start with lemmas before the proof of Theorem 4.1.

Lemma 4.2. *Given a global strong solution $f(\xi, t)$ which is strongly starlike of order < 1 . There exists $\delta' > 0$, such that if $\|\bar{r}_f(0, \cdot)\|_{C^{2, \alpha}(S^1)} < \delta'$, then*

$$\limsup_{t \rightarrow \infty} \|\bar{r}_f(t, \cdot)\|_{C^{2, \alpha}(S^1)}(2t)^\lambda = 0, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right)$$

where $n_0 = \min\{k \geq 1 \mid M_k(f) \neq 0\}$.

Proof. Let $g(\xi, \tau) = \frac{f(\xi, t)}{\sqrt{M_0(0)}}$ where $\tau = \frac{2\pi t}{M_0(0)}$. Then

$$\operatorname{Re} \left[g_\tau \overline{g'} \xi \right] = \frac{1}{2\pi}, \quad \xi \in D \quad \text{and} \quad |g(D, 0)| = \pi.$$

Since the boundary of $g(D, \tau)$ is analytic, then $\bar{r}_g(t, \cdot) \in h^{2, \alpha}(S^1)$ where $h^{2, \alpha}(S^1)$ is the little Hölder space as defined in Vondenhoff [11]. Then by Theorem 3.3 and Theorem 4.3 in Vondenhoff [11], we obtain that there exists $\delta' > 0$ such that if $\|\bar{r}_g(0, \cdot)\|_{C^{2, \alpha}(S^1)} < \delta'$, then

$$\limsup_{\tau \rightarrow \infty} \|\bar{r}_g(\tau, \cdot)\|_{C^{2, \alpha}(S^1)}(2\tau)^\lambda = 0, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right)$$

where $n_0 = \min\{k \geq 1 \mid M_k(f) \neq 0\} = \min\{k \geq 1 \mid M_k(g) \neq 0\}$. Here $\|\bar{r}_f(t, \cdot)\|_{C^{2,\alpha}(S^1)} = \|\bar{r}_g(\tau, \cdot)\|_{C^{2,\alpha}(S^1)}$. Therefore, we conclude that if $\|\bar{r}_f(0, \cdot)\|_{C^{2,\alpha}(S^1)} < \delta'$,

$$\limsup_{t \rightarrow \infty} \|\bar{r}_f(t, \cdot)\|_{C^{2,\alpha}(S^1)} (2t)^\lambda = 0, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right)$$

where $n_0 = \min\{k \geq 1 \mid M_k(f) \neq 0\}$. \square

Lemma 4.3. *Given a global strong degree k_0 polynomial solution $f_{k_0}(\xi, t)$ to (1.1), then there exists $r > 1$ such that for $t \geq 0$,*

$$f_{k_0}(\xi, t) \in O(\overline{D_r}).$$

Also given $\epsilon > 0, T_0 > 0, k \in \mathbb{N}$ and $1 < r' < r$, there exist $\delta(f_{k_0}, T_0, \epsilon, k, r') > 0$ and $\rho(f_{k_0}, T_0, \epsilon, k, r') > 1$ such that if $\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\rho, k} < \delta$ where $f(0, 0) = 0$ and $f'(0, 0) > 0$, then the strong solution $f(\xi, t)$ to (1.1) satisfies

$$f(\xi, t) \in O(\overline{D_{r'}}) \cap C^1([0, T_0], H(D_r)),$$

and for $0 \leq n \leq k, 0 \leq t \leq T_0$,

$$\left| f_{k_0}^{(n)}(\cdot, t) - f^{(n)}(\cdot, t) \right|_{M(r)} < \epsilon.$$

Proof. (a) There exists $r > 1$ such that $f_{k_0}(\xi, t) \in O(\overline{D_r})$ for all $t > 0$.

(b) By Theorem 1.1, we are done with the proof. \square

Lemma 4.4. *Define $M_0\pi$ as the area of $f(D)$ for some $f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i$ in $O(\overline{D})$. Given $\delta' > 0$, there exists $\epsilon' > 0$ such that if $\left| \frac{f^{(j)}}{a_1} \right|_M < \epsilon'$ for $2 \leq j \leq 3$, then $f(\xi)$ is strongly starlike of order < 1 and $\|\bar{r}_f\|_{C^{2,\alpha}(S^1)} < \delta'$ where $\bar{r}_f(\theta) = \frac{|f(\xi)|}{\sqrt{M_0}} - 1$ and $\theta = \arg f(\xi)$.*

Proof. If $\epsilon' < 1$, then $\left| \frac{f''}{a_1} \right|_M < 1$. This implies that $\sum_{n=2}^{\infty} n|a_n| < |a_1|$ which is a sufficient condition for coefficients of strongly starlike functions; see Pommerenke [6].

Now we treat the quantity $\|\bar{r}_f\|_{C^{2,\alpha}(S^1)}$ by calculating $\max_{\theta \in S^1} |\partial_{\theta}^{(j)} \bar{r}_f|, 0 \leq j \leq 3$. Note that the value M_0 can be represented by $a_1^2 + \sum_{n=2}^{\infty} n|a_n|^2$. The function \bar{r}_f satisfies

$$\max_{\theta \in S^1} |\bar{r}_f| \leq \left| \frac{a_1}{\sqrt{M_0}} - 1 \right| + \sum_{n=2}^{\infty} \left| \frac{a_n}{\sqrt{M_0}} \right|$$

which goes to zero as ϵ' goes to zero. The function $\partial_{\theta} \bar{r}_f$ satisfies

$$\max_{\theta \in S^1} |\partial_{\theta} \bar{r}_f| = \max_{\xi \in \partial D} \left| \frac{1}{\operatorname{Re} \left[\frac{f' \xi}{f} \right]} \frac{\operatorname{Im} [\xi f' \bar{f}]}{|f| \sqrt{M_0}} \right|$$

which goes to zero as ϵ' goes to zero. Similarly, $\max_{\theta \in S^1} |\partial_\theta^2 \bar{r}_f|$ and $\max_{\theta \in S^1} |\partial_\theta^3 \bar{r}_f|$ go to zero as ϵ' goes to zero. We conclude that $\|\bar{r}_f\|_{C^{2,\alpha}(S^1)}$ goes to zero as ϵ' goes to zero.

Finally, there exists $0 < \epsilon' < 1$ such that the theorem holds. \square

Proof. (proof of Theorem 4.1)

(a) Denote

$$f(\xi, t) = \sum_{i=1}^{\infty} b_i(t) \xi^i; \quad f_{k_0}(\xi, t) = \sum_{i=1}^{k_0} a_i(t) \xi^i.$$

Note that $b_1^2(t) \geq b_1^2(0) + 2t$ and $a_1^2(t) \geq a_1^2(0) + 2t$ as shown in Kuznetsova [4].

We separate the proof for (a) into (1)-(5) as follows:

- (1) There exists $\delta' > 0$ as stated in Lemma 4.2.
- (2) For such $\delta' > 0$, we can find $\epsilon' > 0$ as stated in Lemma 4.4.
- (3) Given $\epsilon' > 0$, there exists $T_0 > \frac{1}{2}$ such that for $t \geq T_0$,

$$\left| \frac{f_{k_0}^{(2)}(\cdot, t)}{a_1(t)} \right|_M < \frac{1}{8} \epsilon' \quad \text{and} \quad \left| \frac{f_{k_0}^{(3)}(\cdot, t)}{a_1(t)} \right|_M < \frac{1}{8} \epsilon' \quad (4.1)$$

since the coefficients $\{a_i(t)\}_{i \geq 2}$ are bounded and $a_1(t) \geq \sqrt{2t + a_1^2(0)}$ as shown in Kuznetsova [4].

(4) By Lemma 4.3, for such T_0 and ϵ' , there exist $\rho > 1$ and $\delta > 0$ such that if $\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\rho, 3} < \delta$, then

- (i) the strong solution to (1.1) $f(\xi, t)$ exists for $t \in [0, T_0]$, and
- (ii) for $0 \leq t \leq T_0$, $1 \leq j \leq 3$,

$$\left| f_{k_0}^{(j)}(\cdot, t) - f^{(j)}(\cdot, t) \right|_M < \min \left\{ \frac{1}{2} a_1(T_0), \frac{1}{8} \epsilon' \right\}. \quad (4.2)$$

From (4.2) and the fact that $T_0 \geq 1$, we also can obtain that $b_1(T_0) \geq \max\{1, \frac{1}{2} a_1(T_0)\}$. Therefore, by (4.1), (4.2) and the fact that $b_1(T_0) \geq \max\{1, \frac{1}{2} a_1(T_0)\}$, we have

$$\left| \frac{f^{(j)}(\cdot, T_0)}{b_1(T_0)} \right|_M \leq \frac{1}{2} \epsilon', \quad 2 \leq j \leq 3.$$

Due to the fact in (2), $f(\xi, T_0)$ is strongly starlike of order < 1 and

$$\|\bar{r}_f(T_0, \cdot)\|_{C^{2,\alpha}(S^1)} < \delta',$$

where $\bar{r}_f(t, \theta) = \frac{|f(\xi, t)|}{\sqrt{M_0(t)}} - 1$ and $\theta = \arg f(\xi, t)$.

(5) By (1)-(4), we conclude that there exist $T_0 > 0$, $\rho > 1$, $\delta > 0$ such that if $\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\rho, 3} < \delta$, then

- (i) the strong solution $f(\xi, t)$ exists for $t \in [0, T_0]$, and
- (ii) $f(\xi, T_0) \in O(\overline{D})$ is a strongly starlike function of order < 1 , and
- (iii) $\|\bar{r}_f(T_0, \cdot)\|_{C^{2,\alpha}(S^1)} < \delta'$.

By Theorem 2.1 in Gustafsson, Prokhorov and Vasil'ev [2], the solution $f(\xi, t)$ must be global and $f(\xi, t), t \geq T_0$ has strictly decreasing strongly starlike order $\alpha(t)$ since $f(\xi, T_0) \in O(\overline{D})$ and is a strongly starlike function. This also implies that $\overline{r}_f(t, \cdot)$ is well-defined for $t \geq T_0$.

(b) From (5), the assumptions in Lemma 4.2 are satisfied and we obtain

$$\limsup_{T_0 \leq t \rightarrow \infty} \|\overline{r}_f(t, \cdot)\|_{C^{2,\alpha}(S^1)} (2t)^\lambda = 0, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right).$$

□

4.2 Geometric meaning of rescaling behavior in Theorem 4.1

The initial domains we consider in this section are

$$\{f_{k_0}(D, 0) \mid f_{k_0}(\xi, t) \text{ is a global strong polynomial solution of degree } k_0 \in N\}$$

and small perturbations of them. Theorem 4.1 demonstrates that starting with an initial domain $\Omega(0)$ as above, we can obtain a global solution $\Omega(t)$ which is simply connected and has a real analytic boundary, and a rescaling behavior is given in terms of moments. Here we aim to give a geometric characterization for this rescaling behavior by carrying out some explicit calculation:

Theorem 4.5. *Given a global strong solution $f(\xi, t)$ where $f(\xi, 0)$ satisfies the assumption of Theorem 4.1 and $\Omega(t) = f(D, t)$. We show that the rescaled domain $\Omega'(t) = \{x \mid x\sqrt{|\Omega(t)|/\pi} \in \Omega(t)\}$ has radius satisfy that*

$$\max_{z \in \partial\Omega'(t)} ||z| - 1| = o\left(\frac{1}{t}\right)^\lambda, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right)$$

and its curvature $\kappa(t, z), z \in \Omega'(t)$ satisfies

$$\max_{z \in \Omega'(t)} |\kappa(t, z) - 1| = o\left(\frac{1}{t}\right)^\lambda, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right),$$

where $n_0 = \min\{k \geq 1 \mid M_k(f) \neq 0\}$.

Proof. Let $f(\xi, t)$ be a global strong solution satisfies Theorem 4.1. There exists $T_0 > 0$ such that $\overline{r}_f(t, \theta), t \geq T_0$ is well-defined. The value $|\kappa(t, z) - 1|$ satisfies

$$|\kappa - 1| = \left| \frac{(1 + \overline{r}_f)^2 + 2(\overline{r}'_f)^2 - \overline{r}''_f(1 + \overline{r}_f)}{\left[(1 + \overline{r}_f)^2 + (\overline{r}'_f)^2\right]^{\frac{3}{2}}} - 1 \right| = O(\|\overline{r}_f\|_{C^2(S^1)}) \quad (4.3)$$

as $\|\overline{r}_f\|_{C^2}$ approaches 0. Since $\|\overline{r}_f\|_{C^{2,\alpha}(S^1)} = o\left(\frac{1}{t}\right)^\lambda, \forall \lambda \in (0, 1 + \frac{n_0}{2})$ by the results in Theorem 4.1, we can obtain from (4.3) that

$$\max_{z \in \Omega'(t)} |\kappa(t, z) - 1| = o\left(\frac{1}{t}\right)^\lambda, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right).$$

Similarly, since $\|\bar{r}_f\|_{C^{2,\alpha}(S^1)} = o(\frac{1}{t})^\lambda, \forall \lambda \in (0, 1 + \frac{n_0}{2})$ by the results in Theorem 4.1, we can obtain that radius satisfies

$$\max_{z \in \partial\Omega'(t)} ||z| - 1| = o\left(\frac{1}{t}\right)^\lambda, \quad \forall \lambda \in \left(0, 1 + \frac{n_0}{2}\right).$$

□

5 Existence and uniqueness proof of the P-G equation

In this section, we assume the short-time well-posedness of strong* polynomial solutions as shown in Gustafsson [1] and we give a shorter proof of short-time well-posedness for strong solutions in the injection case. Especially, the proof of short-time existence of strong solutions is an application of Theorem 2.4 and this proof implies that every strong solution can be approximated by many strong* polynomial solutions locally in time. The uniqueness proof is given separately.

5.1 Existence

Theorem 5.1. *Given $f(\xi, 0) \in \omega(\overline{D_r}) \cap H(\overline{D_{\rho_0}})$ where $\rho_0 > r > 1$, then there exist $t_0 > 0$ and a strong* solution to (2.1) $f(\xi, t) \in C^1([0, t_0], H(D_r)) \cap \omega(D_r)$ with the initial value $f(\xi, 0)$.*

Proof. (a). For $f(\xi, 0) = \sum_{i=1}^{\infty} a_i(0)\xi^i \in H(\overline{D_{\rho_0}})$, there exists $M > 0$ such that

$$|a_i(0)| \leq M\rho_0^{-i}.$$

Define $f_n(\xi, 0) = \sum_{i=1}^n a_i(0)\xi^i$. Then

$$\left| \min_{\overline{D_r}} |f'(\cdot, 0)| - \min_{\overline{D_r}} |f'_n(\cdot, 0)| \right| \leq \sum_{i=n+1}^{\infty} i |a_i(0)| (r)^i \leq \sum_{i=n+1}^{\infty} i M \left(\frac{\rho_0}{r}\right)^{-i}$$

where $\sum_{i=n+1}^{\infty} i M \left(\frac{\rho_0}{r}\right)^{-i}$ approaches zero as n approaches ∞ . Therefore there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2} \min_{\overline{D_r}} |f'(\cdot, 0)| \leq \min_{\overline{D_r}} |f'_n(\cdot, 0)|, \quad n \geq n_0$$

and $f_n(\xi, 0) \in \omega(\overline{D_r})$. By Gustafsson [1], there exists a strong* polynomial solution $f_n(\xi, t) \in \omega(\overline{D_r})$ at least for a short time.

(b). Given $1 < r_0 < \frac{\rho_0}{r}$, there exists $k_0 \geq n_0$ such that

$$\sum_{k=k_0+1}^{\infty} |a_k(0)| \left(\frac{\rho_0}{r_0}\right)^k k^{3/2} \leq \frac{1}{\sqrt{k_0}} \frac{1}{8} \min_{\overline{D_r}} |f'(\cdot, 0)| \quad (5.1)$$

(c). There exists $t_1 > 0$ such that the strong* solution to (2.1) $f_{k_0}(\xi, t)$ exists for $0 \leq t \leq t_1$ and

$$\min_{(\overline{D_r}, [0, t_1])} |f'_{k_0}| \geq \frac{1}{4} \min_{\overline{D_r}} |f'(\cdot, 0)|.$$

By the above, (5.1) implies

$$\sum_{k=k_0+1}^{\infty} |a_k(0)| \left(\frac{\rho_0}{r_0} \right)^k k^{3/2} \leq \frac{1}{\sqrt{k_0}} \frac{1}{2} \min_{(\overline{D_r}, [0, t_1])} |f'_{k_0}|. \quad (5.2)$$

The inequality (5.2) implies that

$$\|f(\cdot, 0) - f_{k_0}(\cdot, 0)\|_{\frac{\rho_0}{r_0}, 1} \leq \frac{1}{\sqrt{k_0}} \frac{1}{2} \min_{(\overline{D_r}, [0, t_1])} |f'_{k_0}|.$$

(d). By letting $\rho = \frac{\rho_0}{r_0}$ and $l = \frac{1}{2}$, we can see that assumption (A) in Theorem 2.4 is satisfied from (c). By applying Theorem 2.4, the short-time existence is proven. \square

Remark 5.1. The proof also can be applied to the suction case.

If we assume $f(\xi, 0)$ is univalent, then $f(\xi, t)$ we obtained in Theorem 5.1 is also univalent in short time. Therefore, we obtain the following results.

Theorem 5.2. *Given $f(\xi, 0) \in O(\overline{D_r}) \cap H(\overline{D_{\rho_0}})$ where $\rho_0 > r > 1$, then for $1 < r' < r$, there exists $b > 0$ and a strong solution to (1.1) $f(\xi, t) \in C^1([0, b], H(\overline{D_{r'}})) \cap O(\overline{D_{r'}})$ with the initial value $f(\xi, 0)$.*

Since for a given $f(\xi, 0) \in O(\overline{D})$, there exist $1 < r < \rho_0$ such that $f(\xi, 0) \in H(\overline{D_{\rho_0}}) \cap O(\overline{D_r})$, Theorem 5.2 implies the following directly:

Theorem 5.3. *Given $f(\xi, 0) \in O(\overline{D})$, there exists a strong solution to (1.1) $f(\xi, t)$ locally in time.*

5.2 Uniqueness

Theorem 5.4. *Strong solutions to (1.1) are unique.*

Proof. (1) Let $f(\xi, 0) \in O(\overline{D})$. Assume there are two strong solutions f_1, f_2 with the same initial value $f(\xi, 0)$. There exist $1 < r'$ and $b > 0$ such that $f_i(\xi, t) \in O(\overline{D_{r'}})$ for $0 \leq t \leq b$ and $f_i(\xi, t)$ is continuous in $(\overline{D_{r'}}, [0, b])$ for $1 \leq i \leq 2$. Denote

$$M^2 = \max_{i=1,2} \max_{t \in [0, b]} \int_{\partial D_{r'}} |f'_i|^2 d\theta$$

then

$$|\alpha_i(t)| \leq \frac{M}{i} (r')^{-i}, \quad |\beta_i(t)| \leq \frac{M}{i} (r')^{-i}$$

if we denote $f_1(\xi, t) = \sum_{i=1}^{\infty} \alpha_i(t) \xi^i$ and $f_2(\xi, t) = \sum_{i=1}^{\infty} \beta_i(t) \xi^i$.
(2)By (2.8),

$$\begin{aligned} & \left\| \frac{d}{dt} [f_1 - f_2] \right\|_{L^2([0, 2\pi])} \\ & \leq \left\{ \max_{\partial D} \left| P \left[\frac{1}{|f_2'|^2} \right] \right| + C_2 \max_{\partial D} |f_1'| \max_{\partial D} \frac{|f_1'| + |f_2'|}{|f_1'|^2 |f_2'|^2} \right\} \|f_1' - f_2'\|_{L^2([0, 2\pi])}. \end{aligned} \quad (5.3)$$

Therefore, by (5.3), there exists $C > 0$, for $t \in [0, b]$

$$\begin{aligned} & \sum_{i=1}^{\infty} [(\alpha_i - \beta_i)_t]^2 \\ & \leq C \left\{ \sum_{i=1}^{\infty} [(\alpha_i - \beta_i) | i]^2 \right\} \\ & \leq C \left\{ \sum_{i=1}^k [(\alpha_i - \beta_i) | i]^2 + \sum_{i=k+1}^{\infty} (2M)^2 (r')^{-2i} \right\} \\ & \leq C \left\{ \sum_{i=1}^k [(\alpha_i - \beta_i) | i]^2 + 4M^2 \left(\frac{(r')^{-2(k+1)}}{1 - (r')^{-2}} \right) \right\}. \end{aligned}$$

(3)Denote $D_k(t) = \sum_{i=1}^k [(\alpha_i - \beta_i) | i]^2$, then

$$\begin{aligned} D_k'(t) &= \sum_{i=1}^k 2Re \left[(\alpha_i - \beta_i) \overline{(\alpha_i - \beta_i)_t} \right] i^2 \\ &\leq 2k \left\{ \sum_{i=1}^k [(\alpha_i - \beta_i) | i]^2 \right\}^{1/2} \left\{ \sum_{i=1}^k [(\alpha_i - \beta_i)_t]^2 \right\}^{1/2} \\ &\leq 2k C D_k^{1/2}(t) \left\{ D_k(t) + 4M^2 \left(\frac{(r')^{-2(k+1)}}{1 - (r')^{-2}} \right) \right\}^{1/2} \\ &\leq 2k C D_k^{1/2}(t) \left\{ D_k^{1/2}(t) + 2M \left(\frac{(r')^{-(k+1)}}{(1 - (r')^{-2})^{1/2}} \right) \right\} \\ &\leq 2k C D_k(t) + 4k M C D_k^{1/2}(t) \left(\frac{(r')^{-(k+1)}}{(1 - (r')^{-2})^{1/2}} \right). \end{aligned}$$

Note that $|\Omega(t)| = \pi \sum_{i=1}^{\infty} i |\alpha_i(t) - \beta_i(t)|^2 \leq |\Omega(0)| + 2\pi b$ where $|\Omega(t)|$ is the area of the moving domain at time t . So we have $D_k(t) \leq \frac{1}{\pi} 4k |\Omega(t)| \leq \frac{1}{\pi} 4k (|\Omega(0)| + 2\pi b) = 2kA$ for some $A > 0$. Therefore

$$D_k'(t) \leq 2k C D_k(t) + 4M C (2A)^{1/2} k^{3/2} \frac{(r')^{-(k+1)}}{(1 - (r')^{-2})^{1/2}}.$$

Denote $(2A)^{1/2}(4MC)\frac{1}{(1-(r')^{-2})^{1/2}} = C_0$, then

$$\begin{aligned}
D_k'(t) &\leq 2kCD_k(t) + C_0 (r')^{-(k+1)} k^{3/2} \\
(D_k(t)e^{-2kCt})' &\leq e^{-2kCt} C_0 (r')^{-(k+1)} k^{3/2} \\
D_k(t)e^{-2kCt} &\leq \frac{1 - e^{-2kCt}}{2kC} C_0 (r')^{-(k+1)} k^{3/2} \\
D_k(t) &\leq \frac{1}{2kC} (e^{2kCt}) C_0 (r')^{-(k+1)} k^{3/2} = \frac{1}{2r'C} \left(e^{2Ct} (r')^{-1} \right)^k k^{\frac{1}{2}} C_0. \quad (5.4)
\end{aligned}$$

For $0 \leq t < \frac{1}{2C} \ln r'$, in (5.4) we let k approach ∞ , then $D_k(t)$ approaches zero since $\frac{1}{2C}(e^{2Ct}(r')^{-1})^k k^{\frac{1}{2}} C_0$ approaches zero. Therefore $f_1(\xi, t) = f_2(\xi, t)$ for $t \in [0, T)$ where $T = \min\{\frac{1}{2C} \ln r', b\}$.

(4) Hence, the uniqueness of the short-time existence is proven. □

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